Indian Statistical Institute
M. Math. II Year
First Semestral Examination 2006-2007
Representation of Groups

Time: 4hrs | Date: 01-12-2006 | Total Marks: 100

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The paper carries a total of 125 marks. Answer as many questions as you can. But the maximum you may score is 100.

1. Let $\mathcal{K}$ be an algebraically closed field and G be a finite group such that char($\mathcal{K}$) does not divide $\#(G)$. Then define the group algebra $\mathcal{K}[G]$ and prove that it is isomorphic to the self-intertwiner algebra of the left-regular $\mathcal{K}$-representation of G. [25]

2. Let $\mathcal{K}$ be an algebraically closed field and G be a finite group such that char($\mathcal{K}$) does not divide $\#(G)$. Let π be an irreducible $\mathcal{K}$-representation of G and put $\sigma = \pi \oplus \ldots \oplus \pi$ (n copies). Then prove that the self-intertwiner algebra of σ is isomorphic to $\mathcal{K}^{n\times n}$. [25]

3. With the hypothesis as in Question No. (2), let H be a subgroup of G and α be a $\mathcal{K}$-representation of H. Then define the $\mathcal{K}$-representation $\mathrm{Ind}_H^G(\alpha)$ of G and find a formula for its character in terms of the character of H corresponding to α. [20]

4. (a) If χ is an irreducible complex character of a finite group G and C is a conjugacy class of G of size h then show that $\frac{h\cdot\chi(x)}{\chi(1)}$ is an algebraic integer. If, further $(h, \chi(1)) = 1$ then show that for each $x \in C$, either $\chi(x) = 0$ or x is represented by a scalar operator in the representation corresponding to χ.

   (b) State Burnside's $p - q$ Theorem and use part (a) to prove it.

   (In proving part (a), you may assume the following result without proof: If w is a complex root of unity then the field $Q(w)$ does not contain any non-zero algebraic integer of modules strictly less then 1.) [15+10=25]

5. (a) For any $x \in S_n$ let $C(x)$ denote the centralizer of x in S_n. Let y be any transposition in S_n. Show that $\#C(x) \leq \#C(y)$ for all $x \in S_n,\ x \neq$ id. Prove that equality holds here iff either x is also a transposition or if $n = 6$ and x is a fixed-point-free involution.

(b) Show that if an automorphism of S_n maps transpositions to transpositions then it must be inner. Conclude that for $n \neq 6,\ S_n$ has no outer automorphism.

(c) Prove that S_6 has an outer automorphism.

(Hint: For part (b), note that S_n is generated by the $n-1$ transpositions $(1, 2), \ldots, (n-1, n)$ and any automorphism which maps transpositions to transpositions must send these $n-1$ transpositions to $(x_1, x_2), \ldots, (x_{n-1}, x_n)$, when $x_1 x_2 \ldots x_n$ is a permutation of $1, 2, \ldots, n$.) [10+10+10=30]